数理解析研究所講究録
1082 巻 1999 年 93-101

「表現論と非可換調和解析」研究集会

Invariant hyperfunctions on the prehomogeneous vector space acting the group $\mathbf{GL}_n(\mathbb{R}) \times \mathbf{SO}_{p,q}(\mathbb{R})$.

(1998 年 8 月 5 日)

岐阜大学・工学部 (Gifu University)
室 政和 (Masakazu Muro)

Introduction.

In this talk, we shall study the microlocal structure of the complex power of the irreducible relative invarinat $P(x)$ on the prehomogeneous vector space $(\mathbf{GL}_n(\mathbb{R}) \times \mathbf{SO}_{p,q}(\mathbb{R}), M_{n,m}(\mathbb{R}))$. We shall give some result on the exact order of poles with respect to the power parameter (Theorem 2.3) and the exact support of the principal part of the pole (Theorem 2.4). By these theorems, we can construct a suitable basis of the space of singular invariant hyperfunctions on the space $M_{n,m}(\mathbb{R})$ in some cases. The hyperfunctions belonging to the basis are expressed by the coefficients of the Laurent expansion of $|P(x)|^s$, the complex power of the determinant function. We estimate the exact order of the poles of $|P(x)|^s$ and give the exact support of the negative-order coefficients of the Laurent expansion of $|P(x)|^s$.

1 Complex powers of the determinant function.

In this section, we shall explain our problem more precisely, prepare some notions and notations, and state some preliminary known results.

1.1 Some fundamental definitions.

Let m, n be the positive integers with $m > n$. We put $\boldsymbol{V} := M_{n,m}(\mathbb{R})$ be the space of $n \times m$ matrices over the real field $\mathbb{R}$, We define $\mathbf{GL}_n(\mathbb{R})$ (reap. $\mathbf{SL}_n(\mathbb{R})$) to be the general (resp. special) linear group over $\mathbb{R}$. and denote by $\mathbf{GL}_n(\mathbb{R})^+$ the subgroup consisting of elements with positive determinant. Let p, q be positive integers satisfying $p + q = m$. We define $\mathbf{SO}_{p,q}(\mathbb{R}) := \{g \in \mathbf{GL}_m(\mathbb{R})^+ \mid gI_{p,q}{}^tg = I_{p,q}\}$ where $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. *We suppose that $p, q > n$ for a technical reason in this talk.*

Then the real algebraic group $\boldsymbol{G} := \mathbf{GL}_n(\mathbb{R})^+ \times \mathbf{SO}_{p,q}(\mathbb{R})$ operates on the vector space $\boldsymbol{V}$ by the representation

$$\rho(g) : x \longmapsto g_1 \cdot x \cdot {}^tg_2, \tag{1}$$

with $x \in \boldsymbol{V}$ and $g := (g_1, g_2) \in \boldsymbol{G}$.

Irreducible homogeneous relatively invariant polynomials on $\boldsymbol{V}$ are written by a constant multiple of $P(x) := \det(xI_{p,q}{}^tx)$. The polynomial $P(x)$ is a relative invariant corresponding to the character $\chi(g) := \det(g_1)^2$ with respect to the action of $\boldsymbol{G}$,i.e., $P(\rho(g) \cdot x) = \chi(g)P(x)$.

Let $\boldsymbol{S} := \{x \in \boldsymbol{V} \mid \det(xI_{p,q}{}^tx) = 0\}$ and call $\boldsymbol{S}$ a *singular set* of $\boldsymbol{V}$. The open subset $\boldsymbol{V} - \boldsymbol{S}$ decomposes into $(n+1)$ open $\boldsymbol{G}$-orbits

$$\boldsymbol{V}_i := \{x \in M_{n,m}(\mathbb{R}) \mid \mathrm{sgn}(xI_{p,q}{}^tx) = (n-i, i)\}. \tag{2}$$

with $i = 0, 1, \ldots, n$[1]. Here, $\mathrm{sgn}(x)$ for $x \in Sym_n(\mathbb{R})$ is the signature of the quadratic form $q_x(\vec{v}) := {}^t\vec{v} \cdot x \cdot \vec{v}$ on $\vec{v} \in \mathbb{R}^n$.

[1] If $p < n$ or $q < n$, then $\min\{n, q\} \geq i \geq \max\{0, n-p\}$.

1.2 Reduction to the space of symmetric matrices.

The prehomogeneous vector space $\boldsymbol{V} = M_{n,m}(\mathbb{R})$ acting the group $\mathbf{GL}_n(\mathbb{R}) \times \mathbf{SO}_{p,q}(\mathbb{R})$ is closely related to the symmetric matrix space $Sym_n(\mathbb{R})$ acting the group $\mathbf{GL}_n(\mathbb{R})$. We shall explain the relation by using the reduction map ψ.

Let $\boldsymbol{V}_{sym} := Sym_n(\mathbb{R})$ be the space of $n \times n$ real symmetric matrices. Then the $\boldsymbol{G}_{sym} := \mathbf{GL}_n(\mathbb{R})$ acts on the vector space $\boldsymbol{V}_{sym}$ by the representation

$$\rho_{sym}(g) : y \longmapsto g \cdot y \cdot {}^t g, \tag{3}$$

with $y \in \boldsymbol{V}_{sym}$ and $g \in \mathbf{GL}_n(\mathbb{R})$. Put $Q(y) := \det(y)$. Then $Q(y)$ is an irreducible polynomial on $\boldsymbol{V}_{sym}$, and is relatively invariant corresponding to the character $\chi_{sym}(g) := \det(g)^2$ with respect to the action of $g \in \boldsymbol{G}_{sym}$,i.e., $Q(\rho_{sym}(g) \cdot y) = \chi_{sym}(g)^2 Q(y)$. Let $\boldsymbol{S}_{sym} := \{y \in \boldsymbol{V}_{sym} \mid \det(y) = 0\}$ and we call $\boldsymbol{S}_{sym}$ a *singular set* of $\boldsymbol{V}_{sym}$. The open subset $\boldsymbol{V}_{sym} - \boldsymbol{S}_{sym}$ decomposes into $(n+1)$ open $\boldsymbol{G}_{sym}$-orbits

$$\boldsymbol{V}_{sym,i} := \{y \in Sym_n(\mathbb{R}) \mid \operatorname{sgn}(y) = (n-i, i)\}. \tag{4}$$

with $i = 0, 1, \ldots, n$.

Consider the rational map from $\boldsymbol{V} = M_{n,m}(\mathbb{R})$ to $\boldsymbol{V}_{sym} = Sym_n\mathbb{R}$

$$\psi : x \longmapsto y = x I_{p,q} {}^t x. \tag{5}$$

Then we have the following proposition.

Proposition 1.1 (see Ochiai [Och97]). *1. The map ψ gives one to one correspondence between the set of the connected componets of $\boldsymbol{V} - \boldsymbol{S}$ and that of $\boldsymbol{V}_{sym} - \boldsymbol{S}_{sym}$.*

2. Let $\mathbb{C}[\boldsymbol{V}]$ and $\mathbb{C}[\boldsymbol{V}_{sym}]$ be the polynimial rings on $\boldsymbol{V}$ and $\boldsymbol{V}_{sym}$, respectively. We denote by $\mathbb{C}[\boldsymbol{V}]^{\mathbf{SO}_{p,q}(\mathbb{R})}$ the subring of invarinat polynomials under the action of $\mathbf{SO}_{p,q}(\mathbb{R})$ in $\mathbb{C}[\boldsymbol{V}]$. Then we can identify $\mathbb{C}[\boldsymbol{V}]^{\mathbf{SO}_{p,q}(\mathbb{R})}$ and $\mathbb{C}[\boldsymbol{V}_{sym}]$ by the map ψ. In particular, we have $Q(\psi(x)) = P(x)$.

1.3 Complex powers of relative invariants

For a complex number $s \in \mathbb{C}$, we set

$$|Q(y)|_i^s := \begin{cases} |Q(y)|^s & ,\text{if } y \in \boldsymbol{V}_{sym,i}, \\ 0 & ,\text{if } y \notin \boldsymbol{V}_{sym,i}. \end{cases} \tag{6}$$

For a complex number $s \in \mathbb{C}$, we can define a function $|P(x)|_i$ by

$$|P(x)|_i^s := \begin{cases} |P(x)|^s & ,\text{if } x \in \boldsymbol{V}_i, \\ 0 & ,\text{if } x \notin \boldsymbol{V}_i. \end{cases} \tag{7}$$

on $\boldsymbol{V}$. It is a continuous function if the real part $\Re(s) > 0$.

Let $\mathcal{S}(\boldsymbol{V})$ and $\mathcal{S}(\boldsymbol{V}_{sym})$ be the space of rapidly decreasing smooth functions on $\boldsymbol{V}$ and $\boldsymbol{V}_{sym}$,respectively. is convergent if the real part $\Re(s)$ of s is sufficiently large and is meromorphically extended to the whole complex plane. Thus we can regard $|Q(y)|_i^s$ as a tempered distribution with a meromorphic parameter $s \in \mathbb{C}$.

For $\phi(x) \in \mathcal{S}(\boldsymbol{V})$ or $\phi(y) \in \mathcal{S}(\boldsymbol{V}_{sym})$, the integrals

$$Z_i(\phi, s) := \int_{\boldsymbol{V}} |P(x)|_i^s \phi(x) dx, \tag{8}$$

and

$$Z_i(\phi, s) := \int_{\boldsymbol{V}} |Q(y)|_i^s \phi(y) dy, \tag{9}$$

are convergent if the real part $\Re(s) > -1$. Thus we see that $|P(x)|_i^s$ and $|Q(y)|_i^s$ are hyperfunctions with a holomorphic parameter s for $\Re(s) > -1$ and are meromorphically extended to the whole complex plane.

Consequently, we can regard $|P(x)|_i^s$ and $|Q(y)|_i^s$ as tempered distributions with a meromorphic parameter $s \in \mathbb{C}$.

We consider a linear combination of the hyperfunctions defined by

$$P^{[\vec{a},s]}(x) := \sum_{i=0}^{n} a_i \cdot |P(x)|_i^s, \tag{10}$$

and

$$Q^{[\vec{a},s]}(y) := \sum_{i=0}^{n} a_i \cdot |Q(y)|_i^s, \tag{11}$$

with $s \in \mathbb{C}$ and $\vec{a} := (a_0, a_1, \ldots, a_n) \in \mathbb{C}^{n+1}$. Then $P^{[\vec{a},s]}(x)$ and $Q^{[\vec{a},s]}(y)$ are hyperfunctions with a meromorphic parameter $s \in \mathbb{C}$, and depends on $\vec{a} \in \mathbb{C}^{n+1}$ linearly.

1.4 Basic properties and some known results on complex powers.

The following theorem is easily proved by the general theory of b-functions. See for example [Mur90].

Theorem 1.2 (see Muro [Mur98]). *1. $Q^{[\vec{a},s]}(y)$ is holomorphic with respect to $s \in \mathbb{C}$ except for the poles at $s = -\frac{k+1}{2}$ with $k = 1, 2, \ldots$.*

2. The possibly highest order of the pole of $Q^{[\vec{a},s]}(y)$ at $s = -\frac{k+1}{2}$ is given by

$$\begin{cases} \lfloor \frac{k+1}{2} \rfloor & ,(k = 1, 2 \ldots, n-1), \\ \lfloor \frac{n}{2} \rfloor & ,(k = n, n+1 \ldots, \text{ and } k+n \text{ is odd}), \\ \lfloor \frac{n+1}{2} \rfloor & ,(k = n, n+1 \ldots, \text{ and } k+n \text{ is even}). \end{cases} \tag{12}$$

Here, $\lfloor x \rfloor$ means the floor of $x \in \mathbb{R}$, i.e., the largest integer which does not exceed x.

Theorem 1.3. *1. $P^{[\vec{a},s]}(x)$ is holomorphic with respect to $s \in \mathbb{C}$ except for the poles at $s = -\frac{k+1}{2}$ with $k = 1, 2, \ldots$.*

2. The possibly highest order of the pole of $P^{[\vec{a},s]}(x)$ at $s = -\frac{k+1}{2}$ is given by

$$\begin{cases} \lfloor \frac{k+1}{2} \rfloor & ,(1 \le k < n), \\ \lfloor \frac{n}{2} \rfloor & ,(n \le k < m-n \ \&\ k+n \text{ is odd}), \\ \lfloor \frac{n+1}{2} \rfloor & ,(n \le k < m-n \ \&\ k+n \text{ is even}), \\ \lfloor \frac{n}{2} \rfloor + \lfloor \frac{k-m+n+2}{2} \rfloor & ,(m-n \le k < m \ \&\ k+n \text{ is odd}), \\ \lfloor \frac{n+1}{2} \rfloor + \lfloor \frac{k-m+n+2}{2} \rfloor & ,(m-n \le k < m \ \&\ k+n \text{ is even}), \\ \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n+1}{2} \rfloor & ,(m \le k \ \&\ m-n \text{ is even}), \\ 2\lfloor \frac{n}{2} \rfloor & ,(m \le k \ \&\ m-n \text{ is odd and } k+n \text{ is odd}), \\ 2\lfloor \frac{n+1}{2} \rfloor & ,(m \le k \ \&\ m-n \text{ is odd and } k+n \text{ is even}). \end{cases} \tag{13}$$

Here, $\lfloor x \rfloor$ means the floor of $x \in \mathbb{R}$, i.e., the largest integer which does not exceed x.

1.5 Singular invarinat hyperdunctions

We say that a hyperfunction $f(x)$ (resp. $f(y)$) on $\boldsymbol{V}$ (resp. $\boldsymbol{V}_{sym}$) is *singular* if the support of $f(x)$ (resp. $f(y)$) is contained in the set $\boldsymbol{S}$ (resp. $\boldsymbol{S}_{sym}$)). In addition, if $f(x)$ (resp. $f(y)$) is $\boldsymbol{G}^1 := \mathbf{SL}_n(\mathbb{R}) \times \mathbf{SO}_{p,q}(\mathbb{R})$-invariant (resp. $\boldsymbol{G}^1_{sym} := \mathbf{SL}_n(\mathbb{R})$-invariant), we call $f(x)$ (resp. $f(y)$) a *singular invariant* hyperfunction on $\boldsymbol{V}$ (resp. $\boldsymbol{V}_{sym}$).

Any negative-order coefficient of a Laurent expansion of $P^{[\vec{a},s]}(x)$ (resp. $Q^{[\vec{a},s]}(y)$) is a singular invariant hyperfunction since the integral

$$\int f(x)P^{[\vec{a},s]}(x)dx = \sum_{i=0}^{n} Z_i(f,s) \tag{14}$$

$$(resp. \int f(x)Q^{[\vec{a},s]}(y)dy = \sum_{i=0}^{n} Z_i(f,s)) \tag{15}$$

is an entire function with respect to $s \in \mathbb{C}$ if $f(x) \in C_0^\infty(\boldsymbol{V}-\boldsymbol{S})$ (resp. $f(x) \in C_0^\infty(\boldsymbol{V}_{sym}-\boldsymbol{S}_{sym})$), where $C_0^\infty(\boldsymbol{V}-\boldsymbol{S})$ (resp. $C_0^\infty(\boldsymbol{V}_{sym}-\boldsymbol{S}_{sym})$) is the space of compactly supported C^∞-functions on $\boldsymbol{V}-\boldsymbol{S}$ (resp. $\boldsymbol{V}_{sym}-\boldsymbol{S}_{sym}$).

Conversely, we have the following proposition.

Proposition 1.4 ([Mur88],[Mur90]). *Any singular invariant hyperfunction on $\boldsymbol{V}$ (resp. $\boldsymbol{V}_{sym}$) is given as a linear combination of some negative-order coefficients of Laurent expansions of $P^{[\vec{a},s]}(x)$ (resp. $Q^{[\vec{a},s]}(y)$) at various poles and for some $\vec{a} \in \mathbb{C}^{n+1}$.*

1.6 Orbit decomposition of $Sym_n(\mathbb{R})$.

The vector space $\boldsymbol{V}_{sym}$ decomposes into a finite number of $\boldsymbol{G}$-orbits;

$$\boldsymbol{V}_{sym} := \bigsqcup_{\substack{0\le i\le n \\ 0\le j\le n-i}} \boldsymbol{S}^j_{sym,i} \tag{16}$$

where

$$\boldsymbol{S}^j_{sym,i} := \{x \in Sym_n(\mathbb{R}) \mid \mathrm{sgn}(x) = (n-i-j,j)\} \tag{17}$$

with integers $0 \le i \le n$ and $0 \le j \le n-i$. A $\boldsymbol{G}_{sym}$-orbit in $\boldsymbol{S}_{sym}$ is called a *singular orbit*. The subset $\boldsymbol{S}_{sym,i} := \{x \in \boldsymbol{V}_{sym} \mid \mathrm{rank}(x) = n-i\}$ is the set of elements of rank $(n-i)$. It is easily seen that $\boldsymbol{S}_{sym} := \bigsqcup_{1\le i\le n} \boldsymbol{S}_{sym,i}$ and $\boldsymbol{S}_{sym,i} = \bigsqcup_{0\le j\le n-i} \boldsymbol{S}^j_{sym,i}$. Each singular orbit is a stratum which not only is a $\boldsymbol{G}$-orbit but is an $\mathbf{SL}_n(\mathbb{R})$-orbit. The strata $\{\boldsymbol{S}^j_{sym,i}\}_{1\le i\le n, 0\le j\le n-i}$ have the following closure inclusion relation

$$\overline{\boldsymbol{S}^j_{sym,i}} \supset \boldsymbol{S}^{j-1}_{sym,i+1} \cup \boldsymbol{S}^j_{sym,i+1}, \tag{18}$$

where $\overline{\boldsymbol{S}^j_{sym,i}}$ means the *closure* of the stratum $\boldsymbol{S}^j_{sym,i}$.

1.7 Orbit decomposition of $M_{n,m}(\mathbb{R})$.

The vector space $\boldsymbol{V}$ decomposes into a finite number of $\boldsymbol{G}$-orbits;

$$\boldsymbol{V} := \bigsqcup_{\substack{0\le \nu+\mu\le n \\ 0\le \nu,\mu \\ 0\le a\le \nu}} \boldsymbol{S}_{(\nu,\mu;a)} \tag{19}$$

where

$$\boldsymbol{S}_{(\nu,\mu;a)} := \left\{ x \in M_{n,m}(\mathbb{R}) \;\middle|\; \begin{array}{c} \mathrm{rank}(xI_{p,q}{}^t x)=\nu \\ \mathrm{rank}(x)=n-\mu \\ \mathrm{sgn}(xI_{p,q}{}^t x)=(\nu-a,a) \end{array} \right\} \tag{20}$$

with integers $0 \le \nu+\mu \le n$, $0 \le \nu,\mu$ and $0 \le a \le \nu$. Each set $\boldsymbol{S}_{(\nu,\mu;a)}$ is a $\boldsymbol{G}$-orbit in $\boldsymbol{V}$. The codimension of $\boldsymbol{S}_{(\nu,\mu;a)}$ in $\boldsymbol{V}$ is computed easily and it is given by

$$\mathrm{codim}_{\boldsymbol{V}}(\boldsymbol{S}_{(\nu,\mu;a)}) = \mu(m-n+\mu) + (n-\nu-\mu)(\frac{1}{2}(n-\mu-\nu-1)+1) \tag{21}$$

A G-orbit in S is called a *singular orbit*,i.e., an orbit of codimension larger than 1. We denote

$$S_{(\nu,\mu)} := \left\{ x \in M_{n,m}(\mathbb{R}) \;\middle|\; {\scriptstyle \mathrm{rank}(xI_{p,q}{}^t x)=\nu \atop \mathrm{rank}(x)=n-\mu} \right\} = \bigsqcup_{0\le a\le \nu} S_{(\nu,\mu;a)} \tag{22}$$

$$S_{(\nu;a)} := \left\{ x \in M_{n,m}(\mathbb{R}) \;\middle|\; {\scriptstyle \mathrm{rank}(xI_{p,q}{}^t x)=\nu \atop \mathrm{sgn}(xI_{p,q}{}^t x)=(\nu-a,a)} \right\} = \bigsqcup_{0\le \mu\le n-\nu} S_{(\nu,\mu;a)} \tag{23}$$

and

$$S_{(\nu)} := \{x \in M_{n,m}(\mathbb{R}) \mid \mathrm{rank}(xI_{p,q}{}^t x) = \nu\} = \bigsqcup_{0\le \mu\le n-\nu} S_{(\nu,\mu)} \tag{24}$$

In particular, the open orbits are the orbits in $V - S = S_{(n)} = S_{(n,0)}$ and $S_{(n,0,a)} := V_{n-a}$. The singular set S is given by

$$S = \bigsqcup_{0\le\nu\le n-1} S_{(\nu)} = \bigsqcup_{\substack{0\le\nu\le n-1\\ 0\le\mu\le n-\nu}} S_{(\nu,\mu)} = \bigsqcup_{\substack{0\le\nu\le n-1\\ 0\le\mu\le n-\nu \\ 0\le a\le \nu}} S_{(\nu,\mu;a)} \tag{25}$$

Each singular orbit is a stratum which not only is a G-orbit but is an $\mathbf{SL}_n(\mathbb{R}) \times \mathbf{SO}_{p,q}(\mathbb{R})$-orbit. The strata $\{S_{(\nu,\mu;a)}\}_{0\le\nu\le n, 0\le\mu\le n-\nu, 0\le a\le\nu}$ have the following closure inclusion relation

$$\overline{S_{(\nu,\mu;a)}} \supset (S_{(\nu-1,\mu;a)} \cup S_{(\nu-1,\mu;a-1)}) \cup S_{(\nu,\mu+1;a)}, \tag{26}$$

where $\overline{S_{(\nu,\mu;a)}}$ means the *closure* of the stratum $S_{(\nu,\mu;a)}$.

1.8 Relations of singular orbits.

Proposition 1.5 (Relations of singular orbits.). *1. For an open orbit V_i $(i = 1, 2, \ldots, n+1)$ in $V - S$, we have*

$$\psi^{-1}(V_{sym,i}) = V_i = S_{(n,0;i)} \tag{27}$$

2. We have

$$\psi^{-1}(S^j_{sym,i}) = S_{(n-i;j)} = \bigsqcup_{0\le\mu\le n-\nu} S_{(n-i,\mu;j)} \tag{28}$$

3. The largest dimensional orbit in $\psi^{-1}(S^j_i)$ is $S_{(n-i,0;j)}$.

2 Statement of the main results.

In this section we shall give the main problems and results. When we give a complex $(n+1)$ dimensional complex vector $\vec{a} \in \mathbb{C}^{n+1}$, we can determine the exact order of poles of $P^{[\vec{a},s]}(x)$ and the exact support of the hyperfunctions appearing in the principal part of the Laurent expansion. We shall give the statement of the theorems in this section without proofs. Their proofs will be given in §5.

2.1 The problem.

The fundamental question of the study of invariant hyperfunctions obtained as complex powers of relatively invariant polynomials is the following.

Problem 2.1. What are the principal parts of the Laurent expansion of $P^{[\vec{a},s]}(x)$ at poles ? What are their exact orders of poles ? What are the supports of negative-order coefficients of a Laurent expansion of $P^{[\vec{a},s]}(x)$ at poles ?

2.2 Coefficient Vectors.

In order to determine the exact order of the pole of $P^{[\vec{a},s]}(x)$ at $s = s_0$, we introduce the coefficient vectors

$$\boldsymbol{d}^{(k)}[s_0] := (d_0^{(k)}[s_0], d_1^{(k)}[s_0], \ldots, d_{n-k}^{(k)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-k+1} \tag{29}$$

with $k = 0, 1, \ldots, n$. Here, $(\mathbb{C}^{n+1})^*$ means the dual vector space of $\mathbb{C}^{n+1}$. Each element of $\boldsymbol{d}^{(k)}[s_0]$ is a linear form on $\vec{a} \in \mathbb{C}^{n+1}$ depending on $s_0 \in \mathbb{C}$,i.e.,a linear map from $\mathbb{C}^{n+1}$ to $\mathbb{C}$,

$$d_i^{(k)}[s_0] : \mathbb{C}^{n+1} \ni \vec{a} \longmapsto \langle d_i^{(k)}[s_0], \vec{a}\rangle \in \mathbb{C}. \tag{30}$$

We denote

$$\langle \boldsymbol{d}^{(k)}[s_0], \vec{a}\rangle = (\langle d_0^{(k)}[s_0], \vec{a}\rangle, \langle d_1^{(k)}[s_0], \vec{a}\rangle, \ldots, \langle d_{n-k}^{(k)}[s_0], \vec{a}\rangle) \in \mathbb{C}^{n-k+1}. \tag{31}$$

Definition 2.1 (Coefficient vectors $\boldsymbol{d}^{(k)}[s_0]$). Let s_0 be a *half-integer*, i.e., a rational number given by $q/2$ with an integer q. We define the *coefficient vectors* $\boldsymbol{d}^{(k)}[s_0]$ $(k = 0, 1, \ldots, n)$ by induction in the following way.

1. First, we set

$$\boldsymbol{d}^{(0)}[s_0] := (d_0^{(0)}[s_0], d_1^{(0)}[s_0], \ldots, d_n^{(0)}[s_0]) \tag{32}$$

   such that $\langle d_i^{(0)}[s_0], \vec{a}\rangle := a_i$ for $i = 0, 1, \ldots, n$.

2. Next, we define $\boldsymbol{d}^{(1)}[s_0]$ and $\boldsymbol{d}^{(2)}[s_0]$ by

$$\boldsymbol{d}^{(1)}[s_0] := (d_0^{(1)}[s_0], d_1^{(1)}[s_0], \ldots, d_{n-1}^{(1)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n}, \tag{33}$$

   with $d_j^{(1)}[s_0] := d_j^{(0)}[s_0] + \epsilon[s_0] d_{j+1}^{(0)}[s_0]$, and

$$\boldsymbol{d}^{(2)}[s_0] := (d_0^{(2)}[s_0], d_1^{(2)}[s_0], \ldots, d_{n-2}^{(2)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-1}, \tag{34}$$

   with $d_j^{(2)}[s_0] := d_j^{(0)}[s_0] + d_{j+2}^{(0)}[s_0]$. Here,

$$\epsilon[s_0] := \begin{cases} 1 & ,(\text{if } s_0 \text{ is a strict half-integer}), \\ (-1)^{s_0+1} & ,(\text{if } s_0 \text{ is an integer}). \end{cases} \tag{35}$$

   A *strict half-integer* means a rational number given by $q/2$ with an odd integer q.

3. Lastly, by induction on k, we define the coefficient vectors $\boldsymbol{d}^{(k)}[s_0]$ for $k = 0, 1, \ldots, n$ by

$$\boldsymbol{d}^{(2l+1)}[s_0] := (d_0^{(2l+1)}[s_0], d_1^{(2l+1)}[s_0], \ldots, d_{n-2l-1}^{(2l+1)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-2l}, \tag{36}$$

   with $d_j^{(2l+1)}[s_0] := d_j^{(2l-1)}[s_0] - d_{j+2}^{(2l-1)}[s_0]$, and

$$\boldsymbol{d}^{(2l)}[s_0] := (d_0^{(2l)}[s_0], d_1^{(2l)}[s_0], \ldots, d_{n-2l}^{(2l)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-2l+1}, \tag{37}$$

   with $d_j^{(2l)}[s_0] := d_j^{(2l-2)}[s_0] + d_{j+2}^{(2l-2)}[s_0]$.

   Then we have the following proposition.

Proposition 2.1. *Let s_0 be a half-integer. For an integer i in $0 \le i \le n-2$ and $\vec{a} \in \mathbb{C}^{n+1}$, if $\langle \boldsymbol{d}^{(i)}[s_0], \vec{a}\rangle = 0$, then $\langle \boldsymbol{d}^{(i+2)}[s_0], \vec{a}\rangle = 0$. In other words, if $\langle \boldsymbol{d}^{(i+2)}[s_0], \vec{a}\rangle \neq 0$, then $\langle \boldsymbol{d}^{(i)}[s_0], \vec{a}\rangle \neq 0$.*

Proof. This proposition is a trivial consequence from the definition of $\boldsymbol{d}^{(i)}[s_0]$. $\square$

Corollary 2.2. *Let s_0 be a half-integer. Then we have the following results.*

1. *There exists an even integer i_0 in $0 \le i_0 \le n+1$ such that*

$$\langle \boldsymbol{d}^{(i)}[s_0], \vec{a}\rangle \text{ is } \begin{cases} \neq 0 & \text{for all odd } i \text{ in } 0 \le i < i_0. \\ = 0 & \text{for all odd } i \text{ in } n \ge i > i_0. \end{cases} \tag{38}$$

2. *There exists an odd integer i_1 in $-1 \le i_1 \le n+1$ such that*

$$\langle \boldsymbol{d}^{(i)}[s_0], \vec{a}\rangle \text{ is } \begin{cases} \neq 0 & \text{for all even } i \text{ in } 0 \le i < i_1. \\ = 0 & \text{for all even } i \text{ in } n \ge i > i_1. \end{cases} \tag{39}$$

Proof. We can easily prove these corollaries by induction on i. □

2.3 Results on the poles of the complex power functions.

Using the above mentioned vectors $\boldsymbol{d}^{(k)}[s_0]$, we can determine the exact orders of poles of $P^{[\vec{a},s]}(x)$.

Theorem 2.3 (Exact orders of poles). *The exact orders of poles of $P^{[\vec{a},s]}(x)$ at $s = -\frac{k+1}{2}$ with integers in $1 \le k \le m-n-1$ are computed by the following algorithm.*

1. *At $s = -\frac{2u+1}{2}(u = 1, 2, \ldots)$, the coefficient vectors $\boldsymbol{d}^{(k)}[-\frac{2u+1}{2}]$ are defined in Definition 2.1. The exact order $P^{[\vec{a},s]}(x)$ at $s = -\frac{2u+1}{2}(u = 1, 2, \ldots)$ is given in terms of the coefficient vector $\boldsymbol{d}^{(2k)}[-\frac{2u+1}{2}]$.*
   (a) *If $1 \le u \le \frac{n}{2}$, then $P^{[\vec{a},s]}(x)$ has a possible pole of order not larger than u.*
      - *If $\langle \boldsymbol{d}^{(2)}[-\frac{2u+1}{2}], \vec{a}\rangle = 0$, then $P^{[\vec{a},s]}(x)$ is holomorphic, and the converse is true.*
      - *For a fixed integer p in $1 \le p < u$, if $\langle \boldsymbol{d}^{(2p+2)}[-\frac{2u+1}{2}], \vec{a}\rangle = 0$ and $\langle \boldsymbol{d}^{(2p)}[-\frac{2u+1}{2}], \vec{a}\rangle \neq 0$, then $P^{[\vec{a},s]}(x)$ has a pole of order p, and the converse is true.*
      - *Lastly, if $\langle \boldsymbol{d}^{(2u)}[-\frac{2u+1}{2}], \vec{a}\rangle \neq 0$, then $P^{[\vec{a},s]}(x)$ has a pole of order u, and the converse is true.*

   (b) *If $u > \frac{n}{2}$, then $P^{[\vec{a},s]}(x)$ has a possible pole of order not larger than $\lfloor \frac{n}{2} \rfloor$*
      - *If $\langle \boldsymbol{d}^{(2)}[-\frac{2u+1}{2}], \vec{a}\rangle = 0$, then $P^{[\vec{a},s]}(x)$ is holomorphic, and the converse is true.*
      - *For a fixed integer p in $1 \le p < \lfloor \frac{n}{2} \rfloor$, if $\langle \boldsymbol{d}^{(2p+2)}[-\frac{2u+1}{2}], \vec{a}\rangle = 0$ and $\langle \boldsymbol{d}^{(2p)}[-\frac{2u+1}{2}], \vec{a}\rangle \neq 0$, then $P^{[\vec{a},s]}(x)$ has a pole of order p, and the converse is true.*
      - *Lastly, $P^{[\vec{a},s]}(x)$ has a pole of order $\lfloor \frac{n}{2} \rfloor$ if $\langle \boldsymbol{d}^{(n-1)}[-\frac{2u+1}{2}], \vec{a}\rangle \neq 0$ (when n is odd) or $\langle \boldsymbol{d}^{(n)}[-\frac{2u+1}{2}], \vec{a}\rangle \neq 0$ (when n is even), and the converse is true.*

2. *At $s = -u(u = 1, 2, \ldots)$, the coefficient vectors $\boldsymbol{d}^{(k)}[-u]$ are defined in Definition 2.1 with $\epsilon[-u] = (-1)^{-u+1}$. We obtain the exact order at $s = -u(u = 1, 2, \ldots)$ in terms of the coefficient vectors $\boldsymbol{d}^{(2k+1)}[-u]$.*
   (a) *If $1 \le u \le \frac{n}{2}$, then $P^{[\vec{a},s]}(x)$ has a possible pole of order not larger than u.*
      - *If $\langle \boldsymbol{d}^{(1)}[-u], \vec{a}\rangle = 0$, then $P^{[\vec{a},s]}(x)$ is holomorphic, and the converse is true.*
      - *For a fixed integer p in $1 \le p < u$, if $\langle \boldsymbol{d}^{(2p+1)}[-u], \vec{a}\rangle = 0$ and $\langle \boldsymbol{d}^{(2p-1)}[-u], \vec{a}\rangle \neq 0$, then $P^{[\vec{a},s]}(x)$ has a pole of order p, and the converse is true.*
      - *Lastly, if $\langle \boldsymbol{d}^{(2u-1)}[-u], \vec{a}\rangle \neq 0$, then $P^{[\vec{a},s]}(x)$ has a pole of order u, and the converse is true.*

   (b) *If $u > \frac{n}{2}$, then $P^{[\vec{a},s]}(x)$ has a possible pole of order not larger than $\lfloor \frac{n+1}{2} \rfloor$*
      - *If $\langle \boldsymbol{d}^{(1)}[-u], \vec{a}\rangle = 0$, then $P^{[\vec{a},s]}(x)$ is holomorphic, and the converse is true.*

- *For a fixed integer p in $1 \le p < \lfloor \frac{n+1}{2} \rfloor$, if $\langle \boldsymbol{d}^{(2p+1)}[-u], \vec{a} \rangle = 0$ and $\langle \boldsymbol{d}^{(2p-1)}[-u], \vec{a} \rangle \neq 0$, then $P^{[\vec{a},s]}(x)$ has a pole of order p, and the converse is true.*
- *Lastly, $P^{[\vec{a},s]}(x)$ has a pole of order $\lfloor \frac{n+1}{2} \rfloor$ if $\langle \boldsymbol{d}^{(n)}[-u], \vec{a} \rangle \neq 0$ (when n is odd) or $\langle \boldsymbol{d}^{(n-1)}[-u], \vec{a} \rangle \neq 0$ (when n is even), and the converse is true.*

2.4 Results on the supports of the principal parts of Laurent expansions.

The exact support of $P^{[\vec{a},s]}(x)$ is given by the following theorem.

Theorem 2.4 (Support of the singular invariant hyperfunctions). *Suppose that $P^{[\vec{a},s]}(x)$ has a pole of order p at $s = -\frac{k+1}{2}$ with integers k in $1 \le k \le m-n-1$. Let*

$$P^{[\vec{a},s]}(x) = \sum_{w=-p}^{\infty} P_w^{[\vec{a},-\frac{k+1}{2}]}(x)(s + \frac{k+1}{2})^w \tag{40}$$

be the Laurent expansion of $P^{[\vec{a},s]}(x)$ at $s = -\frac{k+1}{2}$. The support of the Laurent expansion coefficients $P_w^{[\vec{a},-\frac{k+1}{2}]}(x)$ is contained in $\boldsymbol{S}$ if $w < 0$.

1. *Let k be an even positive integer. Then the support of $P_w^{[\vec{a},-\frac{k+1}{2}]}(x)$ for $w = -1, -2, \ldots, -p$ is contained in the closure $\overline{S_{-2w}}$. More precisely, it is given by*

$$\mathbf{Supp}(P_w^{[\vec{a},-\frac{k+1}{2}]}(x)) = \overline{(\bigcup_{j \in \{0 \le j \le n+2w \mid \langle d_j^{(-2w)}[-\frac{k+1}{2}], \vec{a} \rangle \neq 0\}} \boldsymbol{S}_{(n+2w,0;j)})}. \tag{41}$$

2. *Let k be an odd positive integer. Then the support of $P_w^{[\vec{a},-\frac{k+1}{2}]}(x)$ for $w = -1, -2, \ldots, -p$ is contained in the closure $\overline{\boldsymbol{S}_{-2w-1}}$. More precisely, it is given by*

$$\mathbf{Supp}(P_w^{[\vec{a},-\frac{k+1}{2}]}(x)) = \overline{(\bigcup_{j \in \{0 \le j \le n+2w+1 \mid \langle d_j^{(-2w-1)}[-\frac{k+1}{2}], \vec{a} \rangle \neq 0\}} \boldsymbol{S}_{(n+2w+1,0;j)})}. \tag{42}$$

Here, **Supp**$(-)$ *means the support of the hyperfunction in* $(-)$.

2.5 Results at the pole at $s = -\frac{k+1}{2}$ with integers k in $m - n \le k$.

The order of poles of the hyperfunction $P^{[\vec{a},s]}(x)$ at $s = -\frac{k+1}{2}$ with $k = m-n, m-n+1, \ldots$ may have poles of order larger than those given by Theorem 2.3. However, by restricting the hyperfunction $P^{[\vec{a},s]}(x)$ to a certain small open set U, Theorem 2.3 and Theorem 2.4 are valid for all poles at $s = -\frac{k+1}{2}$ with $k = 1, 2, \ldots$. We shall expain it.

Proposition 2.5. *Consider the orbit $\boldsymbol{S}_{(0,0)} = \boldsymbol{S}_{(0,0;0)}$ defined by (22) and let x_0 be a point in $\boldsymbol{S}_{(0,0)}$. Then there exists a non-empty open neighborhood of x_0 satisfying*

$$U = \bigsqcup_{\substack{0 \le \nu \le n \\ 0 \le a \le \nu}} (\boldsymbol{S}_{(\nu,0;a)} \cap U). \tag{43}$$

This means that $\boldsymbol{S}_{(\nu,\mu;a)} \cap U = \emptyset$ for all ν and a in $0 \le \nu \le n-\mu, 0 \le a \le \nu$ if $\mu \ge 1$. (On the other hand, we can see easily that $\boldsymbol{S}_{(\nu,0;a)} \cap U \neq \emptyset$ for all ν and a in $0 \le \nu \le n, 0 \le a \le \nu$.) The largest such open set is

$$\boldsymbol{V} - \bigsqcup_{\substack{1 \le \mu \le n \\ 1 \le \nu \le n-\mu \\ 0 \le a \le \nu}} \boldsymbol{S}_{(\nu,\mu;a)}. \tag{44}$$

Then we have the following theorems

Theorem 2.6 (Exact orders of poles 2). *Let U be the open set defined in Proposition 2.5. Then the exact orders of poles of $P^{[\vec{a},s]}(x)|_U$ at $s=-\frac{k+1}{2}$ with $k=1,2,\ldots$ are computed by the algorithms given by Theorem 2.3.*

Theorem 2.7 (Support of the singular invariant hyperfunctions 2). *Suppose that $P^{[\vec{a},s]}(x)_U$ has a pole of order p at $s=-\frac{k+1}{2}$ with $k=1,2,\ldots$. Let*

$$P^{[\vec{a},s]}(x)|_U=\sum_{w=-p}^{\infty}P_w^{[\vec{a},-\frac{k+1}{2}]}(x)|_U(s+\frac{k+1}{2})^w \tag{45}$$

be the Laurent expansion of $P^{[\vec{a},s]}(x)$ at $s=-\frac{k+1}{2}$. The support of the Laurent expansion coefficients $P_w^{[\vec{a},-\frac{k+1}{2}]}(x)|_U$ is contained in $\boldsymbol{S}\cap U$ if $w<0$.

1. *Let k be an even positive integer. Then the support of $P_w^{[\vec{a},-\frac{k+1}{2}]}(x)|_U$ for $w=-1,-2,\ldots,-p$ is contained in the closure $\overline{S_{-2w}}\cap U$. More precisely, it is given by*

$$\mathbf{Supp}(P_w^{[\vec{a},-\frac{k+1}{2}]}(x)|_U)=(\overline{\bigcup_{j\in\{0\le j\le n+2w\ \mid\ \langle d_j^{(-2w)}[-\frac{k+1}{2}],\vec{a}\rangle\neq 0\}}\boldsymbol{S}_{(n+2w,0;j)}})\cap U. \tag{46}$$

2. *Let k be an odd positive integer. Then the support of $P_w^{[\vec{a},-\frac{k+1}{2}]}(x)|_U$ for $w=-1,-2,\ldots,-p$ is contained in the closure $\overline{\boldsymbol{S}_{-2w-1}}\cap U$. More precisely, it is given by*

$$\mathbf{Supp}(P_w^{[\vec{a},-\frac{k+1}{2}]}(x)|_U)=(\overline{\bigcup_{j\in\{0\le j\le n+2w+1\ \mid\ \langle d_j^{(-2w-1)}[-\frac{k+1}{2}],\vec{a}\rangle\neq 0\}}\boldsymbol{S}_{(n+2w+1,0;j)}})\cap U. \tag{47}$$

References

[Mur88] M. Muro, *Singular invariant tempered distributions on regular prehomogeneous vector spaces*, J. Funct. Anal. **76** (1988), no. 2, 317 – 345.

[Mur90] M. Muro, *Invariant hyperfunctions on regular prehomogeneous vector spaces of commutative parabolic type*, Tôhoku Math. J. (2) **42** (1990), no. 2, 163–193.

[Mur98] M. Muro, *Singular Invariant Hyperfunctions on the space of real symmetric matrices*, preprint,(tentatively circulated in RIMS Kokyuuroku No.999,49-91), 1998.

[Och97] H. Ochiai, *Quotients of Prehomogeneous Vector Spaces*, J. Algebra **192** (1997), 61–73, Article No. JA966979.